Logistics

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Hong Kong University

September 1, 2021

- Instructor: Dr.Yu Hao
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- Consultation times: Monday 10:00am – 11:00am
- Tutor: Ms. Karen Mai
- Email: maixt@hku.hk
- Consultation times: Thursday, TBA

- This course studies how practical problems can be solved by applying econometric methods.
- The emphasis is on the application of econometric methods to the analysis of real world economic data using R.
- Topics include multilinear regressions, limited dependent variable, panel data, experiments and quasi-experiments, instrumental variables, time series and forecasting, and basics of machine learning.
- Pre-requisites:
  - Solid knowledge of statistics
  - Introductory level econometric methods
  - Linear algebra

- Students are encouraged to check out student resources the publisher provides on Moodle.
- Lecture slides will be posted on the course website.
- Additional handouts will be distributed when needed to supplement the textbook.

Main textbook:

- Stock, James and Mark Watson (2018), *Introduction to Econometrics*, Global Edition, 4th ed., Pearson. (It is perfectly fine if you use the 3rd edition of this textbook.)
- Hanck, Christoph, Martin Arnold, Alexander Gerber, and Martin Schmelzer. "Introduction to Econometrics with R." University of Duisburg-Essen (2019).

Supplementary textbook:

- Jeffrey Wooldridge , *Introductory Econometrics: A Modern Approach*, Ohio : South-Western Cengage Learning.
- Bruce E. Hansen(2021) , *Introduction to Econometrics*.
- Cunningham, Scott. "Causal Inference." *The Mixtape* 1 (2020).
- Sheppard, Kevin. "Introduction to Python for econometrics, statistics and data analysis." *Self-published, University of Oxford, version* 2 (2012).

- 6 Assignments in total, distributed every Saturday
  - Group work with group of maximum 5 students
  - Graded based on the best 5 assignments
  - Need to submit coding exercises, R notebook is recommended. (Karen will help with setting up)
- Midterm on September 18th.
- Final (time and venue TBA)

- Week 1
  - Introduction: Evidence and Policy
  - Causality and Validity Probability and Statistical Theory Review (SW Ch. 2 -3)
  - Introduction to Programming Language
- Week 2
  - Bivariate Regression (SW Ch. 4 – 5)
  - Multivariate Regression I & II (SW Ch. 6 - 7)
- Week 3
  - Nonlinear Regression models: Quadratic and Logarithms (SW, Ch. 8)
  - Threats to (Internal / External) Validity (SW, Ch. 9)
  - Midterm

- Week 4
  - Regressions with binary dependent variables (SW, Ch. 11)
  - Regressions with panel data (SW Ch. 10; Wooldridge Ch.13-14)
- Week 5
  - Instrumental variables methods (SW, Ch. 12; Wooldridge, Ch. 15)
  - Experiments and Quasi-Experiments (Chapter 13)
- Week 6
  - Times series regressions and forecasting (SW, Ch. 14)
  - Machine Learning (material will be distributed by instructors.)

Review of Probability [1]

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[1] This section is based on Stock and Watson (2020), Chapter 2.

Probabilities and Outcomes

- An **outcomes** is a specific result:
  - Coin toss: either H or T.
  - Roll of dice: $1, 2 \ldots, 6$.
- The **probability** of an **outcome** is the proportion of the time that the outcome occurs in the long run.
  - Fair coin toss: 50 % chance of heads.
- The **sample space** is the set of all possible outcomes.
  - In a coin flip the sample space is $S = \{H, T\}$.
  - If two coins are flipped the sample space is $S = \{HH, HT, TH, TT\}$.
- An **event** is a subset of the sample space.
  - Roll a die $A = \{1, 2\}$.

Probability Function

Definition 1 (Probability Function)

A function $\mathbb{P}$ which assigns a numerical value to events is called a probability function if it satisfies the following Axioms of Probability:

1. $\mathbb{P}(A) \geq 0$.
2. $\mathbb{P}(S) = 1$.
3. If $A_1, A_2, \ldots$ are disjoint then $\mathbb{P}|\cup_{j=1}^{N} A_j| = \sum_{j=1}^{N} \mathbb{P}(A_j)$.

Conditional Probability

Definition 2 (Conditional Probability)

If $\mathbb{P}(B) > 0$, then the **conditional probability** of A given B is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$\mathbb{P}(B)$ is the **marginal probability** of event B.

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B).$$

Joint Events

Take two events H and M.

- let be H the event that an individual's monthly wage exceeds HKD 20000,
- let M be the event that the individual has a master's degree.

Table: Joint Distribution

	Master degree	Non-master degree	Any education
High wage	0.19	0.12	0.31
Low wage	0.17	0.52	0.69
Any wage	0.36	0.64	1.00

Conditional Probability - Example

The probability of earning a high wage conditional on high education is

$$\begin{aligned} & \mathbb{P}(\text{High wage}|\text{Master degree}) \\ = & \frac{\mathbb{P}(\text{High wage} \cap \text{Master degree})}{\mathbb{P}(\text{High wage} \cap \text{Master degree}) + \mathbb{P}(\text{Low wage} \cap \text{Master degree})} \\ = & \frac{0.19}{0.36} = 0.53. \end{aligned}$$

Similarly, the probability of earning a high wage conditional on non-master degree is $\mathbb{P}(\text{High wage}|\text{Non-master degree}) = \frac{0.12}{0.64} = 0.19.$

Independence

- The events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Some facts:

- When events are independent then joint probabilities can be calculated by multiplying individual probabilities.
- If A and B are disjoint then they cannot be independent.

Bayes Rule

Theorem 3 (Bayes Rule)

If $\mathbb{P}(A) > 0$ *and* $\mathbb{P}(B) > 0$ *then*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}.$$

Bayes Rule-Example I

Consider the PCR test for Covid-19:

- If actual infected: 0.9 positive, 0.1 negative.
- If not infected: 0.05 positive, 0.95 negative.
- 1 % population is infected.

What is the chance of being infected given the test is positive?
Let A denote being infected, let B denote test result positive.

- $P(B|A) = 0.9$, $P(B|A^c) = 0.05$, $P(A) = 0.01$.
- $P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)} = \frac{0.9*0.01}{0.9*0.01+0.05*0.99} \approx 0.1538.$

What is the chance of not infected given the test is negative?

- $P(A^c|B^c) = \frac{P(B^c|A^c)P(A^c)}{P(B^c|A)P(A)+P(B^c|A^c)P(A^c)} = \frac{0.95*0.99}{0.1*0.01+0.95*0.99} \approx 0.9989.$

Random Variables

Definition 4 (Random variable)

A random variable is a real-valued outcome; a function from the sample space S to the real line $\mathbb{R}$.

For example, X is a mapping from the coin flip sample space to the real line, with T mapped to 0 and H mapped to 1.

$$X = \begin{cases} 1 & \text{if H} \\ 0 & \text{if T.} \end{cases}$$

Properties of random variables.

- The **expected value** is the long-run average of the random variable.
- The **standard deviation** measures the spread of a probability distribution.

Discrete Random Variables

- The set $\mathcal{X}$ is discrete if it has a **finite or countably infinite number of elements**.
- If there is a discrete set $\mathcal{X}$ such that $\mathbb{P}(X \in \mathcal{X}) = 1$ then X is a discrete random variable.
- The **probability mass function** of a random variable is $\pi(x) = \mathbb{P}(X = x)$, the probability that X equals the value x.
- The **probability distribution** of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur.

Expectation

For a dsicrete variable X with the support of $\mathcal{X}$, the expctation is computed as

$$\mathbb{E}(X) = \sum_{x \in \mathcal{X}} \pi(x)x.$$

- $X = 1$ with the probability of p and $X = 0$ with probability $1 - p$. The expected value is $\mathbb{E}(X) = 1 * p + 0 * (1 - p) = p$.

The expectation of the function of X, $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \pi(x)g(x).$$

Cumulative distribution function

The **cumulative probability distribution(*CDF*)** is the probability that the random variable is less than or equal to a particular value,

$$F(x) = \mathbb{P}(X \leq x),$$

where the probability event is $X \leq x$.

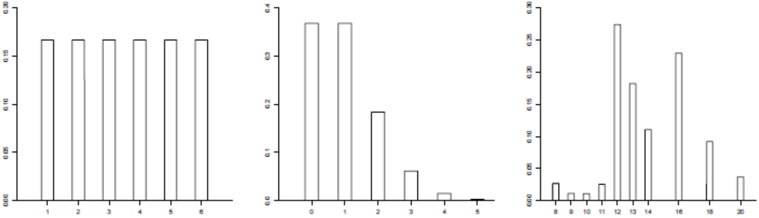
Properties of a CDF If $F(x)$ is a distribution function, then

1. $F(x)$ is non-decreasing.
2. $\lim_{x \to -\infty} F(x) = 0$.
3. $\lim_{x \to \infty} F(x) = 1$.
4. $F(x)$ is right-continuous, $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

Example-Probability mass function

Some examples for discrete variables.

- For a fair dice toss, the support is $\mathcal{X} = \{1, 2 \dots, 6\}$ with the probability mass function is $\pi(x) = \frac{1}{6}$ for $x \in \mathcal{X}$.
- An example of infinite countable random variable is the Poisson distribution, the probability mass function is $\pi(x) = \frac{e^{-1}}{x!}, x = 0, 1, \dots$.



(a) Die Toss (b) Poisson (c) Education

Example-Probability function

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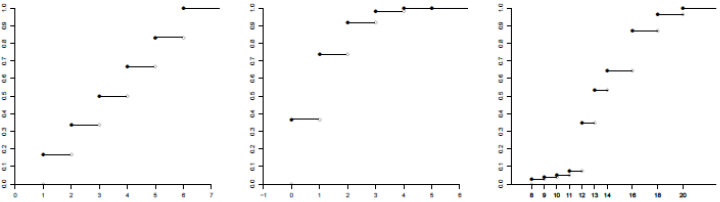


(a) Die Toss (b) Poisson (c) Education

Continuous Random Variables

- The **probability density function(**$p.d.f$**)** area under the probability density function between any two points is the probability that the random variable falls between those two points.
  - the probability for a continous variable to take any value is 0.
  - definition is different from discrete random variables.
- When $F(x)$ is differentiable, the density function is $f(x) = \frac{dF(x)}{dx}$.

Theorem 5 (Properties of density function)

A function $f(x)$ is a density function if and only if

- $f(x) \geq 0 \forall x$.
- $\int_0^\infty f(x)dx = 1$.

Example - Continuous Variables

- Uniform distribution. The CDF is $F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$. The PDF is $f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$.
- Exponential distribution. The CDF is $F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 - \exp(-x) & \text{if} x \geq 0 \end{cases}$. The PDF is $f(x) = \exp(-x), x \geq 0$.

Expectation

If X is a continuous random variable with the density function $f(x)$, its expectation is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

when the integral is convergent.
The expectation of the function of X, $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Some examples:

- $f(x) = 1$ if $0 \le x \le 1$, $\mathbb{E}(X) = \int_0^1 x f(x) = 0.5$.
- $f(x) = \exp(-x)$ if $x \ge 0$,
  $\mathbb{E}(X) = \int_0^{\infty} x \exp(-x) dx = 1$(integration by part).

Mean, variance and Higher Moment

Suppose X is a random variable(either discrete or continous).

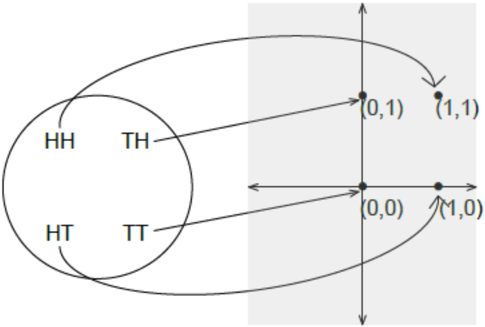
- The **mean** of X is $\mu = \mathbb{E}(X)$.
- The **variance** of X is $\sigma^2 = var(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$.
  - The **standard deviation** of X is the positive root of the variance, $\sigma = \sqrt{\sigma^2}$.
- The $m-th$ **moment** of X is $\mu'_m = \mathbb{E}(X^m)$ and the $m-th$ **central moment** of X is $\mu_m = \mathbb{E}((X - \mathbb{E}(X))^m)$.
  - The **skewness** of X is defined as $skewness = \frac{\mathbb{E}((X-\mathbb{E}(X))^3)}{\sigma^3}$. If the distribution is symmetric, the skewness is 0.
  - The **kurtosis** of X is defined as $skewness = \frac{\mathbb{E}((X-\mathbb{E}(X))^4)}{\sigma^4}$.

Bivariate random variables

A pair of **bivariate random variables** is a pair of numerical outcomes; a function from the sample space to $\mathbb{R}^2$.

A pair of bivariate random variables are typically represented by a pair of uppercase Latin characters such as (X, Y). Specific values will be written by a pair of lower case characters, e.g. (x, y).

Figure: Tossing two coins



Joint distribution functions

The **joint distribution function(Joint CDF)** of (X, Y) is defined as $F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})$.

- A pair of random variables is **discrete** if there is a discrete set $(P) \in \mathbb{R}^2$ such that $\mathbb{P}((X, Y) \in \mathscr{P}) = 1$.
  - The set $\mathcal{P}$ is the support of (X, Y) and consists of a set of points in $\mathbb{R}^2$.
  - The **joint probability mass function** is defined as $p(x, y) = \mathbb{P}(X = x, Y = y)$.
- The pair (X, Y) has a continuous distribution if the joint distribution function $F(x, y)$ is **continuous** in (x, y).
  - When $F(x, y)$ is continuous and differentiable its **joint density (joint PDF)** $f(x, y)$ equals $f(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$.

Bivariate expectation

The **expected value** of real-valued $g(X, Y)$ is

$$\mathbb{E}(g(X, Y)) = \sum_{(x,y)\in\mathbb{R}^2, \pi(x,y)>0} g(x, y)\pi(x, y),$$

and

$$\mathbb{E}(g(X, Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy.$$

Marginal distribution

The **marginal distribution(marginal CDF)** of X is

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y \le \infty) = \lim_{y \to \infty} F(x, y).$$

- ▷ In the continuous case,

$$F_X(x) = \lim_{y \to \infty} \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) du dv = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(u, v) du dv.$$

The **marginal densities(marginal PDF)** of X is the derivative of the marginal CDF of X,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(u, v) du dv = \int_{-\infty}^{\infty} f(x, y) dy.$$

Conditional distributions I

The conditional cumulative distributions:

- The **conditional distribution function** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \mathbb{P}(Y \leq y|X = x)$$

  for any x such that $\mathbb{P}(X = x) > 0$, If X has a discrete distribution.

- For continous X, Y, the **conditional distribution** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \lim_{\epsilon \downarrow 0} \mathbb{P}(Y \leq y|x - \epsilon \leq X \leq x + \epsilon).$$

The conditional density:

- For continous variable (X, Y), the conditional density function (conditional PDF) is defined by $f_{Y|X}(y|x) = \frac{d}{dy} F_{Y|X}(y|x)$.

Independence I

- Recall that two events A and B are independent if the probability that they both occur equals the product of their probabilities, thus $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- Consider the events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$.
- The probability that they both occur is $\mathbb{P}(A \cap B) = \mathbb{P}(X \leq x, Y \leq y) = F(x, y)$.
- If $F(x, y) = F_X(x)F_Y(y)$ then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- The random variables X and Y are **statistically independent** if for all x, y, $F(x, y) = F_X(x)F_Y(y)$.
- If X, Y have differentiable density function, X, Y are statistically independent if $f(x, y) = f_X(x)f_Y(y)$.
- The discrete random variables X and Y are **statistically independent** if for all x, y, $\pi(x, y) = \pi_X(x)\pi_Y(y)$.

Independence II

- If X, Y are independent and continuously distributed, then

$$f_{Y|X}(y|x) = f(y),$$
$$f_{X|Y}(x|y) = f(x).$$

- If X and Y are independent then for any functions, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}|g(X)| < \infty$ and $\mathbb{E}|h(Y)| < \infty$, then

$$\mathbb{E}(g(X)h(Y)) = \mathbb{E}_X(g(X))\mathbb{E}_Y(h(Y)).$$

Covariance and correlation I

- If X and Y have finite variances, the **covariance** between X and Y is

$$cov(X,Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

- The **correlation** between X and Y is

$$corr(X,Y) = \frac{cov(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

Covariance and correlation II

- If X and Y are independent with finite variances, then X and Y are uncorrelated.
  - The reverse is not true. For example, suppose that $X \sim U[-1, 1]$. Since it is symmetrically distributed about 0 we see that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^3] = 0$. Set $Y = X^2$. Then $cov(X, Y) = \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X] = 0$. Thus X and Y are uncorrelated yet are fully dependent!
- If X and Y have finite variances, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2cov(X, Y)$.
- If X and Y are independent, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

Conditional Expectation

Just as the expectation is the central tendency of a distribution, the conditional expectation is the central tendency of a conditional distribution.

The **conditional expectation(conditional mean)** of Y given $X = x$ is the expected value of the conditional distribution $F_{Y|X}(y|x)$ and is written as $\mathbb{E}(Y|X = x)$.

- For discrete random variables, it is defined as

$$\mathbb{E}(Y|X = x) = \frac{\sum_y y\pi(x, y)}{\pi_X(x)}.$$

- For continuous random variables, it is defined as

$$\mathbb{E}[Y|X = x] = \frac{\int_y yf(x, y)}{f_X(x)}.$$

Law of Iterated Expectations I

- Consider $m(X) = \mathbb{E}[Y|X]$ a transformation of X.
- We can take expectation with respect to $m(X)$

Theorem 6 (Law of Iterated Expectations(LIE))

If $\mathbb{E}[Y] < \infty$, *then* $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

Univariate Normal

Definition 7 (Standard Normal Dist.)

A random variable Z has the **standard normal distribution**, write $Z \sim N(0,1)$, if it has the density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), x \in \mathbb{R}.$$

Note the standard normal density is typically written as $\phi(x)$. The CDF does not have closed form but is written as $\Phi(x)$.
If $X \sim N(\mu, \sigma^2)$ and $\sigma > 0$ then X has the density

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}), x \in \mathbb{R}.$$

Multivariate Normal I

Let $Z_1, \ldots, Z_m$ be i.i.d $N(0, 1)$. The joint density is the product of the marginal densities:

$$\begin{aligned} f(x_1, \ldots, x_m) &= f(x_1) \ldots f(x_m) \\ &= \frac{1}{(2\pi)^{m/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{m} x_i^2\right). \end{aligned}$$

Multivariate Normal II

Let $\mathbf{Z} = [Z_1, \ldots, Z_m]^\top$ be an m-component random vector following standard normal distribution $\mathbf{Z} \sim N(0, \mathbf{I}_m)$ and $\mathbf{X} = \mu + \mathbf{BZ}$ for $q \times m$ matrix $\mathbf{B}$, then $\mathbf{X}$ has the **multivariate normal distribution**, written as $\mathbf{X} \sim N(\mu, \Sigma)$, where $\Sigma = \mathbf{BB}^\top$.
The PDF of $\mathbf{X}$ is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{q/2}(\det \Sigma)^{1/2}} \exp\left(-\frac{(x-\mu)^\top \Sigma^{-1}(x-\mu)}{2}\right).$$

Multivariate Normal III

Properties of multivariate normal distributions.

1. Any linear combination of $X_1, \ldots, X_m$ is normally distributed.
2. The marginal distribution of each random variable is normal.
3. If the covariance of X_1 and X_2 is 0, then X_1 and X_2 are independent. The reverse is true.
4. If X_1 and X_2 are normally distributed with the joint density of $f(x_1, x_2)$, then the marginal expectation of X_1 given X_2 is a linear function of X_2: $\mathbb{E}[X_1|X_2 = x_2] = a + bx_2$.

χ-squared, t and F distribution

- The **chi-squared distribution** is the distribution of the sum of m squared independent standard normal distributed variables.
- Let $\mathbf{Z} \sim N(0, \mathbf{I}_m)$ be multivariate standard normal, then $\mathbf{Z}^\top \mathbf{Z} \sim \chi^2_m$.
- If $\mathbf{X} \sim N(0, \Sigma)$ with Σ positive definite, then $\mathbf{X}^\top \Sigma^{-1} \mathbf{X} \sim \chi^2_m$.
- Let $Q_m \sim \chi^2_m$ and $Q_r \sim \chi^2_r$ be independent. Then $\frac{Q_m/m}{Q_r/r} \sim F_{m,r}$.
- Let $Z \sim N(0,1)$ and $Q_m \sim \chi^2_m$ be independent, then $\frac{Z}{\sqrt{Q_m/m}}$ follows the t-distribution with m degree of freedom, t_m.

Sampling I

Almost all the statistical and econometric procedures used in this text involve averages or weighted averages of a sample of data.

- ▷ Characterizing the distributions of sample averages.
- ▷ The act of random sampling is a random variable, its distribution is **sampling distribution**.

Sampling II

1. **Simple random sampling**. e.g. A commuting student record daily traffic time.
2. **i.i.d. draws**. e.g. $Y_1, \ldots, Y_n$ are randomly drawn from the same population.

i.i.d I

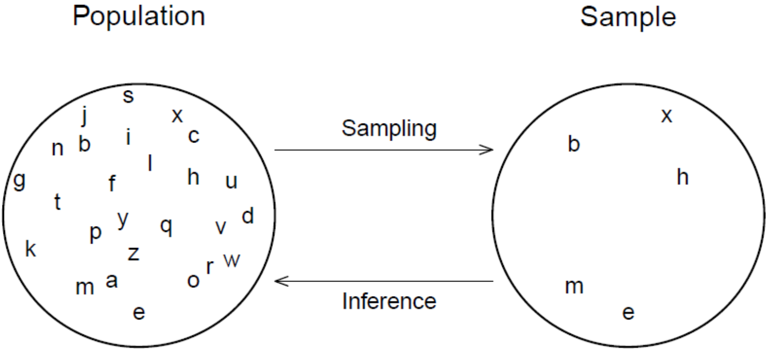
Definition 8 (independent and identically distributed(i.i.d))

The collection of random vectors $\{X_1, \ldots, X_n\}$ are **independent and identically distributed(i.i.d)** if they are mutually independent with identical marginal distributions.

- A collection of random vectors $\{X_1, \ldots, X_n\}$ is a **random sample** from the population F if X_i are i.i.d with distribution F.
- The distribution F is called the **population distribution**. We refer to the distribution as the **data generating process(DGP)**.
- The **sample size** n is the number of individuals in the sample.

i.i.d II

Figure: Sampling and Inference



There are two approaches to characterizing sampling distributions:

1. **the exact approach** derives: the **exact distribution** or **finite-sample distribution** of Y.
2. **the approximate approach** uses approximations to the sampling distribution that rely on the sample size being large, often called the asymptotic distribution.

Two useful tools when the sample size is large: **the law of large numbers** and the **central limit theorem**.

- The law of large numbers says that when the sample size is large, $\bar{Y}$ will be close to μ_Y with very high probability.
- The central limit theorem says that when the sample size is large, the sampling distribution of the standardized sample average, $(\bar{Y} - \mu_Y)/\sigma_{\bar{Y}}$, is approximately normal.

- The sample average, $\bar{Y}$, varies from one randomly chosen sample to the next and thus is a random variable with a sampling distribution. If $Y_1, \ldots, Y_n$ are i.i.d., then
- the sampling distribution of Y has mean μ_Y and variance $\sigma^2_{\bar{Y}} = \sigma^2_Y / n$.
- the law of large numbers says that $\bar{Y}$ converges in probability to μ_Y; and
- the central limit theorem says that the standardized version of $\bar{Y}$, $(\bar{Y} - \mu_Y)/\sigma_{\bar{Y}}$, has a standard normal distribution $N(0,1)$ distribution. when n is large.

Convergence

- A sequence of random variables $Z_n \in \mathbb{R}$ **converges in probability** to c as $n \to \infty$, denoted by $Z_n \to_p c$ or $plim_{n \to \infty} Z_n = c$, if for all $\delta > 0$,

$$\lim_{n \to \infty} \mathbb{P}(|Z_n - c| \leq \delta) = 1.$$

Law of Large Numbers I

Theorem 9 (Weak Law of Large Numbers(WLLN))

If X_i are i.i.d. and $\mathbb{E}(X) < \infty$, then as $n \to \infty$,

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \to_p \mathbb{E}(X).$$

An estimator $\hat{\theta}$ of a parameter θ is **consistent** if $\hat{\theta} \to_p \theta$ as $n \to \infty$.

Continuous Mapping Theorem

Theorem 10 (Continuous Mapping Theorem)

If $Z_n \to_p c$ as $n \to \infty$ and $h(\cdot)$ is continuous at c then $h(Z_n) \to_p h(c)$ as $n \to \infty$.

Central Limit Theorem I

Theorem 11 (Central Limit Theorem(CLT))

If X_i are i.i.d. and $\mathbb{E}(X^2) < \infty$ then as $n \to \infty$

$$\sqrt{n}(\bar{X}_n - \mu) \to_d N(0, \sigma^2),$$

where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{E}[(X - \mu)^2]$.

Slutsky's Theorem I

Theorem 12 (Slutsky's Theorem)

If $Z_n \to_d Z$ *and* $c_n \to_p c$ *as* $n \to \infty$, *then*

1. $Z_n + c_n \to_d Z + c$
2. $Z_n c_n \to_d Zc$
3. $\frac{Z_n}{c_n} \to_d \frac{Z}{c}$ *if* $c \neq 0$.

References I

Stock, J. H. and Watson, M. W. (2020). *Introduction to econometrics*, volume 4. Pearson New York.

Review of Statistics [1]

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[1] This section is based on Stock and Watson (2020), Chapter 3.

Estimators

Suppose you want to understand the distribution of X in the population.

- When a statistic $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ is a function of an i.i.d. sample, then the distribution is determined by the population distribution is F and the sample size is n.
- We call the distribution of $\hat{\theta}$ the **sample distribution**.

The goal of an estimator $\hat{\theta}$ is to learn about the parameter θ, we evaluate the

- The exact bias and variance.
- The distribution under normality.
- The asymptotic distribution as $n \to \infty$.

Goodness of Estimators

Let $\hat{\theta}$ be an estimator of θ. Then

- The bias of $bias(\hat{\theta})$ is $E[\hat{\theta}] - \theta$.
  - We say an estimator is **unbiased** if the bias is 0.
- The **mean squared error** of an estimator $\hat{\theta}$ for θ is

$$mse(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2].$$

  - The mean squared error is $mse(\hat{\theta}) = \text{var}(\hat{\theta}) + (bias(\hat{\theta}))^2$.

Best Unbiased Estimator

Definition 1 (Best Linear Unbiased Estimator (BLUE))

If $\sigma^2 < \infty$ the sample mean $\bar{X}_n$ has the lowest variance among all linear unbiased estimators of μ.

Bias, consistency, and efficiency

- Suppose $Y_1, \ldots, Y_n$ are i.i.d.
- Denote an estimator for μ_Y as $\hat{\mu}_Y$.
- The bias of $\hat{\mu}_Y$ is $E(\hat{\mu}_Y) - \mu_Y$.
- $\hat{\mu}_Y$ is an **unbiased** estimator of $\hat{\mu}_Y$ if $E(\hat{\mu}_Y) = \mu_Y$.
- $\hat{\mu}_Y$ is an **consistent** estimator of $\hat{\mu}_Y$ if $\hat{\mu}_Y \rightarrow_p \mu_Y$.
- Let $\tilde{\mu}_Y$ denote another estimator for μ_Y, and suppose both $\bar{\mu}_Y$ and $\tilde{\mu}_Y$ are consistent. Then $\hat{\mu}_Y$ is more efficient if $var(\hat{\mu}_Y) < var(\tilde{\mu}_Y)$.

Properties

- $E(\bar{Y}) = \mu_Y$, so $\bar{Y}$ is an unbiased estimator of μ_Y.
- the law of large numbers states that $\bar{Y} \rightarrow_p \mu_Y$, $\bar{Y}$ is consistent.
- Consider $\tilde{Y} = \frac{1}{n}(\frac{1}{2}Y_1 + \frac{3}{2}Y_2 + \ldots)$, then $var(\tilde{Y}) = 1.25\sigma_Y^2/n > var(\bar{Y}) = \sigma_Y^2/n$. $\bar{Y}$ is more efficient than $\tilde{Y}$.
- $\bar{Y}$ is the Best Linear Unbiased Estimator for μ_Y.

Hypothesis

- A point hypothesis is the statement that θ equals a specific value θ_0.
- A common example is θ measures the effect the proposed policy. A typical question is whether $\theta = \theta_0$.
- The **null hypothesis**, written as $H_0 : \theta = \theta_0$.
- The **alternative hypothesis**, written as $H_A : \theta \neq \theta_0$, is the set $\{\theta \in \Theta : \theta \neq \theta_0\}$.
    - **One-sided** hypothesis: $H_A : \theta > \theta_0$.
    - **Two-sided** hypothesis: $H_A : \theta \neq \theta_0$.

Acceptance and Rejection

- A hypothesis test is a decision based on data. We can either **fail to reject** the null hypothesis or **reject** the null hypothesis.
- An alternative way to express a decision rule is to construct a real-valued function of the data called a **test statistics**

$$T = T(X_1, \ldots, X_n)$$

  together with a **critical region** C.
- A hypothesis can be expressed as
  - Fail to reject H_0 if $T \in C$.
  - Reject H_0 if $T \notin C$.

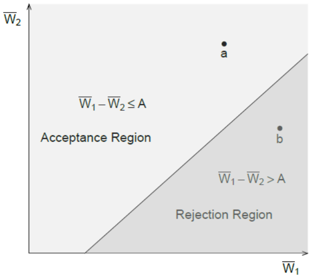
Note: "Accept" H_0 does not mean H_0 is true.

Example - Hypothesis Testing I

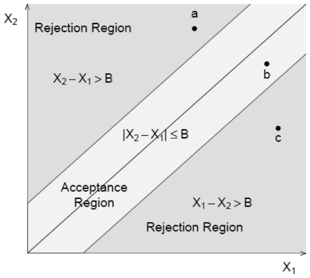
Consider the following examples:

- $2n$ adults who were raised in similar settings, n attended early childhood education. Let $\bar{W}_1$ be the average wage in the early childhood education group, and let $\bar{W}_2$ be the average wage in the remaining sample. Null hypothesis $H_0 : \bar{W}_1 > \bar{W}_2$.
- You ride each bus once and record the time it takes to travel from home to the university. Let X_1 and X_2 be the two recorded travel times. You adopt the following decision rule: If the absolute difference in travel times is greater than B minutes you will reject the hypothesis that the average travel times are the same, otherwise you will accept the hypothesis.

Example - Hypothesis Testing II



(a) Early Childhood Education Example



(b) Bus Travel Example

Type I and Type II error

- A false rejection of the null hypothesis is a **Type I error**.
- A false acceptance of the alternative hypothesis is a **Type II error**.

	Accept H_0	Reject H_0
H_0 true	Correct Decision	Type I Error
H_1 true	Type II Error	Correct Decision

P-value I

- the sample average $\bar{Y}$ will rarely be exactly equal to the hypothesized value $\mu_{Y,0}$.
- Differences between $\bar{Y}$ and $\mu_{Y,0}$ can arise because
  - the true mean is not $\mu_{Y,0}$ (the null hypothesis is false) or
  - the true mean equals $\mu_{Y,0}$ (the null hypothesis is true) but $\bar{Y}$ differs from $\mu_{Y,0}$ because of random sampling.
- impossible to distinguish between these two possibilities with certainty.

P-value II

With a sample of data

- cannot conclude if H_0 Is true.
- can do **probabilistic calculation** that permits testing the null hypothesis in a way that accounts for sampling uncertainty.
- How?compute the p-value of the null hypothesis.

P-value III

- The p-value, also called the **significance probability**, is the probability of drawing a statistic **at least as adverse to the null hypothesis** as the one you actually computed in your sample, assuming the null hypothesis is correct.
- In the case at hand, the p-value is the probability of drawing $\bar{Y}$ at least as far in the tails of its distribution under the null hypothesis as the sample average you actually computed.
- $p-\text{value} = Pr(|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|)$

Example I

In a sample of recent college graduates, the average wage is \$22.64. The p-value is the probability of observing a value of Y at least as different from \$20 (the population mean under the null hypothesis) as the observed value of \$22.64 by pure random sampling variation, assuming that the null hypothesis is true.

- If this p-value is small (say, 0.1%), unlikely that this sample drawn if the null hypothesis is true;
  - reasonable to conclude that the null hypothesis is not true.
- if this p-value is large (say, 40%), likely that the observed sample average of \$22.64 could have arisen just by random sampling variation if the null hypothesis is true;
  - the evidence against the null hypothesis is weak in this probabilistic sense,(**fail to reject**)

Computing p-value

The p-value is the area in the tails of the distribution of $\bar{Y}$ under the null hypothesis beyond $\mu_{Y,0} \pm |\bar{Y}^{act} - \mu_{Y,0}|$.

- To compute the p-value, need to know the shape of the distribution.
- With CLT, the sampling distribution of $\bar{Y}$ is well approximated by a normal distribution

When σ_Y is known, then we can compute the *p*-value

- Recall: By CLT, $(\bar{Y} - \mu_Y)/\sqrt{\sigma_{\bar{Y}}} \rightarrow_d N(0,1)$, then $\sqrt{n}(\bar{Y} - \mu_Y) \rightarrow_d N(0, \sigma_Y^2)$.
- Under the null hypothesis, $p - \text{value} = Pr(\left|\frac{\bar{Y} - \mu_{Y,0}}{\sigma_{\bar{Y}}}\right| > \left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_{\bar{Y}}}\right|) = 2\Phi(-\left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_{\bar{Y}}}\right|)$,
- where Φ is the standard normal cumulative distribution function.

Issue: $\sigma_{\bar{X}}$ **unknown.**

Sample Variance

If the following assumptions hold:

1. $X_1, \ldots, X_n$ are i.i.d.
2. $\mathbb{E}(X_i) < \infty$.

The sample variance is computed

$$\bar{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

- μ is unknown, need to be estimated.
- $\mathbb{E}((X - \bar{X})^2) \to \frac{n-1}{n}\sigma$.
- The sample variance is a consistent estimator of the population variance.

Sample Variance - Example

- When $Y_1, \ldots, Y_n$ are i.i.d. draws from a Bernoulli distribution with success probability p,
- the formula for the variance of $\bar{Y}$ simplifies to $p(1-p)/n$,
- The formula for the standard error also takes on a simple form that depends only on Y and n: $SE(\bar{Y}) = \sqrt{\bar{Y}(1-\bar{Y})} > n$.

t-statistic

The standardized sample average can be constructed using

$$t = \frac{\bar{X} - \mu}{\sqrt{\bar{s}^2}}.$$

With the sample of $x_1, \ldots, x_n$, we can compute the sample t-statistic t^{sample}.
The p-value is given by

$$p\text{-value} = 2\Phi(-|t^{sample}|).$$

Significance Level

When construct hypothesis test, can fix a significance level.

- α-significance test means the tolerance to make Type I error is α.
- α is referred to as the **size** of the test.

Suppose the two-sided test has the **significance level** of α, the rule is "**Reject** H_0 **if** $|t^{sample}| > 1 - \Phi^{-1}(\alpha/2)$".

- $\alpha = 1\%$, $1 - \Phi^{-1}(\alpha/2) = 2.58$.
- $\alpha = 5\%$, $1 - \Phi^{-1}(\alpha/2) = 1.96$.
- $\alpha = 10\%$, $1 - \Phi^{-1}(\alpha/2) = 1.64$.

Confidence Interval I

- random sampling error makes it impossible to learn the exact value of the population mean
- it is possible to **construct a set** of values that contains the true population mean with a certain prespecified probability.
- It's called **confidence set**, the prespecified probability that μ_Y is contained in this set is called the **confidence level**.
- The confidence set for μ_Y turns out to be all the possible values of the mean between a lower and an upper limit, so that the confidence set is an interval, called a **confidence interval**.
- The **coverage probability** of a confidence interval for the population mean is the probability, computed over all possible random samples, that it contains the true population mean.

Confidence Interval II

A 95% two-sided confidence interval for mY is an interval constructed so that it contains the true value of mY in 95% of all possible random samples. When the sample size n is large, 90%, 95%, and 99% confidence intervals for mY are:

- 90% confidence interval for $\mu_Y = \bar{Y} \pm 1.64 SE(\bar{Y})$
- 95% confidence interval for $\mu_Y = \bar{Y} \pm 1.96 SE(\bar{Y})$
- 99% confidence interval for $\mu_Y = \bar{Y} \pm 2.58 SE(\bar{Y})$

Example

- consider the problem of constructing a 95% confidence interval for the mean hourly earnings of recent college graduates using a hypothetical random sample of 200 recent college graduates where
- $\bar{Y} = \$22.64$ and $se(\bar{Y}) = 1.28$.
- The 95% confidence interval for mean hourly earnings is $22.64 \pm 1.96 \times 1.28 = (20.13, 25.15)$.
- The **coverage probability** of a confidence interval for the population mean is the probability, computed over all possible random samples, that it contains the true population mean

Confidence Interval(More general sense) I

We are interested in learning a parameter of interest θ from i.i.d. random sample of $X_1, \ldots, X_n$.

- With random sampling error, it's impossible to learn the exact value of the parameter of interest.
- Construct a **confidence set**: the parameter of interested has $1-\alpha$ probability to fall into the confidence set.
- The **coverage probability** of the interval estimator is the probability that the random interval contains the true parameter.
  - An $1-\alpha$ **asymptotic confidence interval** for a parameter has the **asymptotic coverage probability** $1-\alpha$.

Confidence Interval(More general sense) II

A normal-based $1-\alpha$ confidence interval is

$$CI = [\hat{\theta} - Z_{1-\alpha/2}s(\hat{\theta}), \hat{\theta} + Z_{1-\alpha/2}s(\hat{\theta})],$$

where $\hat{\theta}$ is the estimator for θ and $se(\hat{\theta})$ is the estimated standard deviation. $Z_{1-\alpha/2}$ is the $1-\alpha/2$-quantile of a normal distribution.

Test for Difference Between Two Groups I

Suppose we observe the i.i.d sample $W_1, \ldots, W_{n_1}, \ldots, W_n$.

- Sample $W_1, \ldots, W_{n_1}$ are the monthly wage of graduates with master's degree, let μ_1 denote the population mean and σ_1^2 the population variance of group 1.
- Sample $W_{n_1+1}, \ldots, W_n$ are the monthly wage of graduates with bachelor's degree, let μ_2 denote the population mean and σ_2^2 the population variance of group 2.
- Let $n_2 = n - n_1$.
- $H_0 : \mu_1 - \mu_2 > d_0$, $H_1 : \mu_1 - \mu_2 \leq d_0$, with significance level of α.

Test for Difference Between Two Groups II

- The parameter of interest is $\theta = \mu_1 - \mu_2$.
- Let $\bar{W}_1$ and $\bar{W}_2$ be the estimated sample mean and s_1^2 and s_2^2 be the estimated sample variance for group 1 and group 2.
- The standard error of $\hat{\theta} = \bar{W}_1 - \bar{W}_2$ is $se(\hat{\theta}) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$.
- We construct the t-statistic as $t = \frac{\hat{\theta} - d_0}{se(\hat{\theta})}$.
- We reject H_0 if $t > Z_{1-\alpha}$.

Social Class or Education? Childhood Circumstances and Adult Earning I

This example is based on the example in SW2020 Chapter 3, p.p. 122.

TABLE 3.1 Differences in Household Income According to Childhood Socioeconomic Circumstances, Grouped by Level of Highest Qualification

	Father's NS-SEC = Higher			Father's NS-SEC = Routine			Difference, Higher vs. Routine			
Qualification	$\bar{Y}_h$	s_h	n_h	$\bar{Y}_r$	s_r	n_r	$\bar{Y}_h - \bar{Y}_r$	$SE(\bar{Y}_h - \bar{Y}_r)$	95% Confidence Interval for d	
None	£2,223.13	£2,115.12	1129	£1,842.98	£1,48729	6383	£380.15	£65.64	£251.38	£508.93
GCSE/O-Level	£2,83718	£1,819.73	1962	£2,596.93	£1,738.47	4042	£240.25	£49.35	£143.49	£33700
A-Level	£3,045.99	£2,451.81	1216	£2,745.70	£1,912.50	1169	£300.30	£89.85	£124.11	£476.49
Undergraduate degree or more	£3,690.51	£2,743.55	4359	£3,370.96	£2,443.58	2505	£319.55	£64.11	£193.86	£445.23
All categories	£3,215.71	£2,49773	8666	£2405.45	£1,886.86	14099	£810.25	£31.18	£749.13	£871.38

Source: Understanding Society.

Figure

Social Class or Education? Childhood Circumstances and Adult Earning II

- breaks down the differences in mean household income for individuals according to their are father's NS-SEC occupation type,
- and considers these differences for selected highest level of educational qualification
- The data shows that within both groups according to the NS-SEC of a father's occupation, those with higher qualifications are part of households with higher total income.
- Test the differences between mean income by the father's occupational categorization $(Y_h - Y_r)$ for each of the educational groupings.

Social Class or Education? Childhood Circumstances and Adult Earning III

- For individuals with no qualifications
- test statistics $= \frac{(2223.13-1842.98)}{\sqrt{2115.12^2/1129+1487.29^2/6383}} = 5.7911.$
- The 95 \% CI for the difference $(Y_h - Y_r)$ is $(2223.13 - 1842.98) \pm 1.96\sqrt{2115.12^2/1129 + 1487.29^2/6383} = (251.38, 508.93).$

Use t-distribution when n is small

Consider the t-statistic used to test the hypothesis $H_0 : \mu_Y = \mu_{Y,0}$, using data $Y_1, \dots, Y_n$.

$$t = \frac{\bar{Y} - \mu_{Y,0}}{\sqrt{s_Y^2/n}},$$

where $s_Y^2$is the estimated sample mean.

- When n is larger, under general conditions the t-statistic has a standard normal distribution if the sample size is large and the null hypothesis is true.
- When n is small, then the t-statistic in Equation has a Student t distribution with n - 1 degrees of freedom.

Example I

To illustrate a test for the difference between two means

- let μ_w be the mean hourly earnings in the population of women recently graduated from college,
- let μ_m be the population mean for recently graduated men.
- Consider the null hypothesis that mean earnings for these two populations differ by a certain amount, say, d_0. Then the null hypothesis and the two-sided alternative hypothesis are $H_0 : \mu_m - \mu_w = d_0$ vs. $H_1 : \mu_m - \mu_w \neq d_0$

Example II

Population means are unknown:

- must be estimated from samples of men and women.
- Suppose we have samples of n_m men and n_w women drawn at random from their populations.
- Let the sample average annual earnings be $\bar{Y}_m$ for men and $\bar{Y}_w$ for women.
- An estimator of $\mu_m - \mu_w$ is $\bar{Y}_m - \bar{Y}_w$.

Example III

In asymptotics:

- $\bar{Y}_m \to_d N(\mu_m, \sigma_m^2/n_m)$, $\bar{Y}_w \to_d N(\mu_, \sigma_w^2/n_w)$
- By properties of random distributions, $\bar{Y}_m - \bar{Y}_w \to N(\mu_m - \mu_w, (\sigma_m^2/n_m) + (\sigma_w^2/n_w)^2)$.

Example IV

Construct t-statistics:

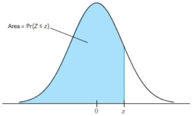
$t = \frac{\bar{Y}_m - \bar{Y}_w}{\sqrt{(\sigma_m^2/n_m)+(\sigma_w^2/n_w)^2}}$.

- When n is larger(> 30), $t \rightarrow_d N(0,1)$.
  - If the test is at 5% significance level, reject if $t > 1.64$.
- When n is small(≤ 30), $t \sim t - \text{dist}_{n-1}$

Distribution Table I

Appendix

TABLE 1 The Cumulative Standard Normal Distribution Function, $\Phi(z) = \Pr(Z \le z)$



z	Second Decimal Value of z: 0	1	2	3	4	5	6	7	8	9
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611

Distribution Table II

z	Second Decimal Value of z									
	0	1	2	3	4	5	6	7	8	9
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

This table can be used to calculate $\Pr(Z \leq z)$ where Z is a standard normal variable. For example, when $z = 1.17$, this probability is 0.8790, which is the table entry for the row labeled 1.1 and the column labeled 7.

References I

Stock, J. H. and Watson, M. W. (2020). *Introduction to econometrics*, volume 4. Pearson New York.